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DESIGNING TWO FACTOR EXPERIMENTS FOR SELECTING INTERACTIONS.(U)
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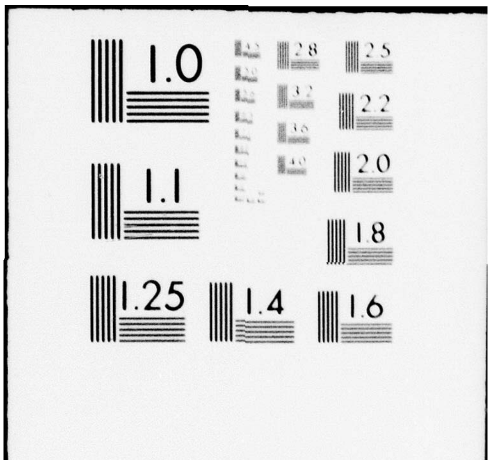

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6 DESIGNING TWO FACTOR EXPERIMENTS
FOR SELECTING INTERACTIONS

by

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1. Introduction

Factorial experiments in which the additive model applies have received special attention in the applied literature because of the simplicity of their analysis and subsequent interpretation. However the recent emphasis in the life sciences on synergism and antagonism (see e.g. Rothman (1974)) have served to highlight the importance of experiments in which interactions are not merely nuisance parameters but the quantities of primary interest. This point of view suggests a number of interesting problems. For example, Neymann (1977) describes a class of multiple comparison problems for determining synergistic effects. This paper pursues a slightly different approach and considers the problem of selecting the treatment combination in a two factor experiment with both factors qualitative for which the corresponding population interaction is a maximum. The problem was introduced by Bechhofer et al. (1977); they give a detailed analysis of the 2×c case (two levels of the first factor and c levels of the second factor). The present work expands their study by considering the design problem for arbitrary r×c experiments when the "natural" procedure based on the sample interactions is used.

In Section 2 the model is introduced and the design requirement is stated for the problem. In Sections 3 and 4 the infimum of the probability of correct selection is studied by reducing the problem to a nonlinear programming problem. The example of designing a 3×4 experiment is studied in Section 5 and the last section considers an alternative (strengthened) version of the design requirement.

2. Model and Design Requirement

A two-factor experiment, with both factors qualitative, is to be performed with the first factor being studied at r levels and the second factor at c levels. The usual fixed-effects linear model is assumed so that observations Y_{ijk} $(1 \leq i \leq r,\ 1 \leq j \leq c,\ 1 \leq k \leq n)$ are independent and normally distributed with means $E[Y_{ijk}] = \mu_{ij} \equiv \mu + \alpha_i + \beta_j + \gamma_{ij}$ where $\sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0$ are the usual identifiability constraints and $\mathrm{Var}(Y_{ijk}) = \sigma^2 < \infty$. Here σ^2 is assumed known but the μ_{ij} are unknown. This paper considers the problem of designing an experiment to select the largest (algebraic) interaction. Let

$$\gamma_{[1]} \leq \cdots \leq \gamma_{[rc]}$$

denote the ordered values of the $\{\gamma_{ij}\}$. It is assumed that the experimenter has no prior knowledge of the pairing of the $\gamma_{[\ell]}$, $1 \leq \ell \leq rc$ and the γ_{ij}. The goal is to select the treatment combination associated with $\gamma_{[rc]}$. Initially the indifference zone formulation of Bechhofer (1954) will be employed and attention will be restricted to procedures which satisfy the following probability (design) requirement:

$$P_{\underset{\sim}{\mu}}[CS] \geq P^* \tag{2.1}$$

whenever $\gamma_{[rc]} \geq \Delta^*$ and $\gamma_{[rc]} - \gamma_{[rc-1]} \geq \delta^*$ where the event $[CS]$ occurs iff the treatment combination corresponding to $\gamma_{[rc]}$ is selected and the constants δ^*, Δ^* and P^* satisfy $0 < \delta^*, \frac{(r-1)(c-1)}{(r-1)(c-1)-1}\delta^* < \Delta^*$ and $\frac{1}{rc} < P^* < 1$.

Intuitively (2.1) requires that a correct selection occur with high probability only when both (a) $\gamma_{[rc]}$ and $\gamma_{[rc-1]}$ are sufficiently far apart and (b) $\gamma_{[rc]}$ is sufficiently positive. When $\gamma_{[rc]} = 0$ the additive model holds. The exact choice of δ^*, Δ^*, P^* depends on economic considerations and is not discussed here. The restrictions on the δ^* and Δ^* values are required for there to exist $\{\gamma_{ij}\}$ matrices which satisfy the conditions of (2.1) i.e. for the problem to be nonempty.

The following "natural" procedure, P, based on the BLUE's, $\hat{\gamma}_{ij}$, of the γ_{ij} will be employed.

P: Take n observations on each treatment combination and compute $\hat{\gamma}_{ij} \equiv \overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...}$ $(1 \leq i \leq r,\ 1 \leq j \leq c)$ where a dot replacing a subscript indicates an average has been computed over the elements for that subscript. Select the treatment combination producing $\hat{\gamma}_{[rc]} \equiv \max\{\hat{\gamma}_{ij}: 1 \leq i \leq r,\ 1 \leq j \leq c\}$ as the one associated with $\gamma_{[rc]}$.

Given $2 \leq r \leq c$, $3 \leq c$ and $(\delta^*, \Delta^*, P^*)$, our problem is to find the smallest value of n which will guarantee (2.1) when P is used.

<u>Remark 2.1</u>. While procedure P is intuitively appealing, its optimality properties are unknown. In particular it is interesting to note that a most economical property for P and its other decision theoretic properties cannot be determined from the works of Hall (1959), Eaton (1967) and Lehmann (1966) since the joint distribution of the $\hat{\gamma}_{ij}$ is singular.

3. Infimum of the Probability of a Correct Selection

As the first step in determining the minimum sample size to achieve (2.1) an expression will be derived for the probability of a correct selection (PCS) when using $\mathcal{P}$ for arbitrary true $\underset{\sim}{\mu}$.

$$P_{\underline{\mu}}[CS|\mathcal{P}] = P[\hat{\gamma}_{11} > \hat{\gamma}_{ab} \ \forall \ (a,b) \neq (1,1)]$$

$$(3.1) \qquad = P\begin{bmatrix} \sum_{i=2}^{r} \sum_{j=2}^{c} \hat{\gamma}_{ij} - \hat{\gamma}_{ab} \geq 0, \quad 2 \leq a \leq r, \ 2 \leq b \leq c; \\ \sum_{i=2}^{r} \sum_{j=2}^{c} \hat{\gamma}_{ij} + \sum_{j=2}^{c} \hat{\gamma}_{aj} \geq 0, \quad 2 \leq a \leq r; \\ \sum_{i=2}^{r} \sum_{j=2}^{c} \hat{\gamma}_{ij} + \sum_{i=2}^{r} \hat{\gamma}_{ib} \geq 0, \quad 2 \leq b \leq c \end{bmatrix}$$

since the $\hat{\gamma}_{ij}$ sum to zero over rows and columns

$$(3.2) \qquad = P[\underset{\sim}{X} + \underset{\sim}{\gamma} \in C] \equiv f_n(\underset{\sim}{\gamma}), \text{ say, where}$$

$$(3.3) \qquad X_{ij} = \hat{\gamma}_{ij} - \gamma_{ij} \quad (1 \leq i \leq r, \ 1 \leq j \leq c),$$

$$(3.4) \qquad \underset{\sim}{X}' = (X_{22},\ldots,X_{2c},X_{32},\ldots,X_{rc}),$$

$$(3.5) \qquad \underset{\sim}{\gamma}' = (\gamma_{22},\ldots,\gamma_{rc}) \text{ and}$$

C is a convex cone defined by (3.1) of the form $\{w \in R^{(r-1)(c-1)} | Aw \geq 0\}$. $\underset{\sim}{X}$ has a nonsingular $(r-1)(c-1)$ variate normal distribution with mean vector zero and covariance matrix Σ given by

$$
\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 & \cdots & \Sigma_2 \\ \Sigma_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Sigma_2 \\ \Sigma_2 & \cdots & \Sigma_2 & \Sigma_1 \end{pmatrix} \tag{3.6}
$$

where $\Sigma_1 = (\sigma^1_{ij})$ and $\Sigma_2 = (\sigma^2_{ij})$ are $(c-1)\times(c-1)$ matrices with entries

$$
\sigma^1_{ij} = \begin{cases} (r-1)(c-1)/rcn & i = j \\ -(r-1)/rcn & i \neq j \end{cases} \tag{3.7}
$$

and

$$
\sigma^2_{ij} = \begin{cases} -(c-1)/rcn & i = j \\ 1/rcn & i \neq j \end{cases} \tag{3.8}
$$

and each row and column of Σ contains $(r-1)$ blocks of Σ_ℓ's.

So the PCS depends on $\underset{\sim}{\mu}$ only through $\underset{\sim}{\gamma}$ and can be written as

$$
\begin{aligned} f_n(\underset{\sim}{\gamma}) &= P[\underset{\sim}{X} \in C - \underset{\sim}{\gamma}] \\ &= \int_{C-\gamma} K \exp\{-\tfrac{1}{2}\underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{x}\} d\underset{\sim}{x} \end{aligned} \tag{3.9}
$$

where $K = (2\pi)^{-(r-1)(c-1)/2}[\det(\Sigma)]^{-1/2}$.

<u>Remark 3.1</u>. It can be easily checked from (3.1) and (3.2) that $f_n(\underset{\sim}{\gamma})$ is constant under row and/or column permutations of the matrix

$$(3.10) \qquad \begin{pmatrix} \gamma_{22} & \cdots & \gamma_{2c} \\ \vdots & & \vdots \\ \gamma_{r2} & \cdots & \gamma_{rc} \end{pmatrix}$$

formed from the vector $\underset{\sim}{\gamma}$.

The design requirement (2.1) will be satisfied if n is the smallest integer satisfying $f_n(\underset{\sim}{\gamma}^0) \geq P^*$ where $\underset{\sim}{\gamma}^0 = \underset{\sim}{\gamma}^0(n)$ is chosen so that

$$(3.11) \qquad f_n(\underset{\sim}{\gamma}^0) = \min_{\underset{\sim}{\gamma} \in F} f_n(\underset{\sim}{\gamma}) \quad \text{and}$$

$F = F(r-1, c-1, \delta^*, \Delta^*)$ is the convex polytope given by

$$\left\{ x = \begin{pmatrix} x_{22} \cdots x_{2c} \\ \vdots \\ x_{r2} \cdots x_{rc} \end{pmatrix} \in R^{(r-1)\times(c-1)} \,\middle|\, \sum_i \sum_j x_{ij} \geq \Delta^*; \right.$$

$$\sum_i \sum_j x_{ij} - x_{ab} \geq \delta^*, \quad 2 \leq a \leq r,\ 2 \leq b \leq c;$$

$$\sum_i \sum_j x_{ij} + \sum_j x_{aj} \geq \delta^*, \quad 2 \leq a \leq r;$$

$$\left. \sum_i \sum_j x_{ij} + \sum_i x_{ib} \geq \delta^*, \quad 2 \leq b \leq c \right\}.$$

In the above notation row and column labels (a and b) run from 2 to r and 2 to c respectively thus preserving the analogy with (3.10). The indices i and j range from 2 to r and 2 to c respectively.
A solution $\underset{\sim}{\gamma}^0$ to the nonlinear programming problem (3.11) is called the

least favorable configuration (LFC) since it represents the worst case the experimenter must design against.

Since the multivariate normal density is log concave and the expression (3.9) for the PCS only involves $\underset{\sim}{\gamma}$ as a location parameter in the domain of integration, it follows directly from the preservation theorem of C. Borell (1973) (see also Rinott (1976)) that $f_n(\underset{\sim}{\gamma})$ is also log concave in $\underset{\sim}{\gamma}$. Hence for $\underset{\sim}{\gamma}_1, \underset{\sim}{\gamma}_2 \in F$ and $0 \leq \alpha \leq 1$ $f_n(\alpha\underset{\sim}{\gamma}_1 + (1-\alpha)\underset{\sim}{\gamma}_2) \geq \{f_n(\underset{\sim}{\gamma}_1)\}^{\alpha}\{f_n(\underset{\sim}{\gamma}_2)\}^{1-\alpha} \geq \min\{f_n(\underset{\sim}{\gamma}_1), f_n(\underset{\sim}{\gamma}_2)\}$ and so the LFC must occur at an extreme point of F.

4. Extreme Point Analysis

First note that $F(r-1,c-1,\delta^*,\Delta^*)$ is given by

$$\Big\{x \in R^{(r-1)\times(c-1)} \Big| \sum_i \sum_j x_{ij}/\delta^* \geq \Delta^*/\delta^*;$$

$$\sum_i \sum_j x_{ij}/\delta^* - x_{ab}/\delta^* \geq 1, \quad 2 \leq a \leq r, \ 2 \leq b \leq c;$$

$$\sum_i \sum_j x_{ij}/\delta^* + \sum_j x_{aj}/\delta^* \geq 1, \quad 2 \leq a \leq r;$$

$$\sum_i \sum_j x_{ij}/\delta^* + \sum_i x_{ib}/\delta^* \geq 1, \quad 2 \leq b \leq c \Big\}$$

and hence $F(r-1,c-1,\delta^*,\Delta^*) = \delta^* \cdot F(r-1,c-1,1,\Delta^*/\delta^*)$ where if α is a scalar and $S \subset E^n$ then $\alpha \cdot S = \{\alpha x | x \in S\}$. So it suffices to determine the extreme points of $F(r-1,c-1,1,\Delta^*)$ in order to solve the extreme point problem for general F since x^* is an extreme point of $F(r-1,c-1,\delta^*,\Delta^*) \Leftrightarrow x^*/\delta^*$ is an extreme point of $F(r-1,c-1,1,\Delta^*/\delta^*)$.

Next note that corresponding to each extreme point x^* of $F(r-1,c-1,1,\Delta^*)$ there are $(r-1)!(c-1)!$ row/column permutation versions of x^* all of which are also extreme points of $F(r-1,c-1,1,\Delta^*)$ since $F(r-1,c-1,1,\Delta^*)$ is row/column permutation invariant. Now from Remark 3.1, $f_n(\underset{\sim}{\gamma})$ is constant over each row/column permutation version of $\underset{\sim}{\gamma}$ and hence the LFC $\underset{\sim}{\gamma}^0$ must be one of the points of the subset $F' = F'(r-1,c-1,1,\Delta)$ of F given by

$$\text{(4.1)} \qquad \{x \in F | x_{22} \geq x_{23} \geq \cdots \geq x_{2c} \text{ and } x_{22} \geq x_{32} \geq \cdots \geq x_{r2}\}$$

$$= \{x \in R^{(r-1)\times(c-1)} | Bx \geq b\}, \quad \text{say.}$$

Here B and k are of dimensions $(rc+r+c-4)\times(r-1)(c-1)$ and $(rc+r+c-4)\times 1$ respectively.

One final simplification of the problem can be made. Choose P orthogonal so that $P\Sigma P'$ is diagonal with entries the eigenvalues of Σ, say $\lambda_1,\ldots,\lambda_{(r-1)(c-1)}$. Then after the change of variables $y = Px$, $f_n(\underset{\sim}{\gamma})$ becomes

$$(4.2) \qquad f_n(\underset{\sim}{\gamma}) = \int_{C'-P\underset{\sim}{\gamma}} \cdots \int K \exp\left\{-\frac{1}{2} \sum_{i=1}^{(r-1)(c-1)} \lambda_i y_i^2\right\} d\underset{\sim}{y}$$

and $C' = \{w \in R^{(r-1)(c-1)} | AP'w \geq 0\}$. Problem (3.11) reduces to determining $\underset{\sim}{\gamma}^0$ so that

$$(4.3) \qquad f_n(\underset{\sim}{\gamma}^0) = \min_{\underset{\sim}{\gamma}\in F'} f_n(\gamma).$$

For moderate r and c there are several efficient algorithms available for computing the entire set of extreme points of F' such as Balinski (1961) and Chernikova (1965). These algorithms systematically select a subsystem of the defining inequality system $Bx \geq b$, solve it in equality form, check the resulting solution for feasibility in the original system and then pivot to a new subsystem. There is an alternative method for determining the extreme points of F' based on some ideas from graph theory. It will be illustrated in the next section where the complete solution to the extreme point problem is determined for the 3×4 case for arbitrary δ^* and Δ^*.

While the enumeration of the extreme points of F' is always computationally feasible, the same cannot be said about evaluating the objective function $f_n(\underset{\sim}{\gamma})$. The expression (4.2) for $f_n(\underset{\sim}{\gamma})$ requires an (r-1)(c-1) dimensional integration over the convex polytope, $C'-P\underset{\sim}{\gamma}$. In the 3×4 example of Section 5, $f_n(\underset{\sim}{\gamma})$ is approximated by a simulation technique based on its probability interpretation (3.1). In summary, until quadrature or other techniques are developed for accurately evaluating high dimensional integrals over arbitrary convex domains, the limiting factor in the implementation of the results of this paper will be the computability of $f_n(\underset{\sim}{\gamma})$.

5. An Example

In Bechhofer et al. (1977) a detailed study is made of the design problem for $2 \times c$ experiments; closed form expressions are obtained for the extreme points of $F'(1,c-1,\delta^*,\Delta^*)$ for arbitrary δ^*, Δ^* and c. In addition the 3×3 case was analyzed. Tables of the sample sizes required to implement P for 2×3, 2×4 and 3×3 experiments are contained in Bechhofer and Santner (1978).

The design problem will now be studied for the 3×4 experiment for arbitrary $\delta^* > 0$ and $\Delta^* > 1.2\delta^*$. First the extreme points of $F'(\Delta^*) \equiv F'(2,3,1,\Delta^*)$ over $\Delta^* > 1.2$ must be determined. Since it is desired to leave Δ^* arbitrary, the algorithms mentioned in Section 4 cannot be used to solve the problem. Instead the following two-pronged strategy will be used. First the set of extreme points of $F'(\Delta^*)$ above the face $\sum_i \sum_j x_{ij} = \Delta^*$ will be determined and then the set of extreme points of $F'(\Delta^*)$ on the face $\sum \sum x_{ij} = \Delta^*$ will be determined.

The extreme points of $F'(\Delta^*)$ satisfying $\sum_i \sum_j x_{ij} > \Delta^*$ can be found by computing the vertices of $F'(2,3,1,1.2)$. Now $F'(2,3,1,1.2) \subset F'(\Delta^*)$ since $\Delta^* > 1.2$ and hence every vertex of $F'(2,3,1,1.2)$ with $\sum_i \sum_j x_{ij} > \Delta^*$ will be a vertex of $F'(\Delta^*)$ above the face $\sum_i \sum_j x_{ij} = \Delta^*$. An application of Balinski's algorithm yields:

$$\begin{pmatrix} .5 & .5 & .5 \\ .5 & .5 & -1 \end{pmatrix} \quad \text{with } \sum_i \sum_j x_{ij} = 1.5 \text{ and} \tag{5.1}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \quad \text{with } \sum_i \sum_j x_{ij} = 2. \tag{5.2}$$

Hence for $1.2 < \Delta^* \le 1.5$ both (5.1) and (5.2) are extreme points of $F'(\Delta^*)$ while for $1.5 < \Delta^* \le 2$ only (5.2) is an extreme point of $F'(\Delta^*)$ and finally for $2 < \Delta^* < \infty$ neither (5.1) nor (5.2) is an extreme point of $F'(\Delta^*)$.

Next the set of extreme points of $F'(\Delta^*)$ on the face $\sum_i \sum_j x_{ij} = \Delta^*$ will be found. The analysis of $F''(\Delta^*) \equiv F'(\Delta^*) \cap \{x \in R^{2\times 3} | \sum \sum x_{ij} = \Delta^*\}$ is more easily conducted on its isomorphic image, T, under the transformation ϕ given by

$$\begin{pmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} \Delta^* & \Delta^*-1 & \Delta^*-1 & \Delta^*-1 \\ \Delta^*-1 & \Delta^*-1 & \Delta^*-1 & \Delta^*-1 \\ \Delta^*-1 & \Delta^*-1 & \Delta^*-1 & \Delta^*-1 \end{pmatrix}$$

$$- \begin{pmatrix} \sum \sum x_{ij} & -\sum x_{i2} & -\sum x_{i3} & -\sum x_{i4} \\ -\sum x_{2j} & x_{22} & x_{23} & x_{24} \\ -\sum x_{3j} & x_{32} & x_{33} & x_{34} \end{pmatrix}.$$

It can easily be seen that T is given by

$$\left\{ x = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix} \middle| \; x_{ij} \ge 0 \quad \forall (i,j) \varepsilon \right.$$

$$\sum_j x_{1j} = 4\Delta^*-3; \quad \sum_j x_{aj} = 4\Delta^*-4, \quad a = 2,3;$$

$$\sum_i x_{i1} = 3\Delta^*-2; \quad \sum_i x_{ib} = 3\Delta^*-3, \quad b = 2,3,4;$$

$$\left. x_{22} \le x_{23} \le x_{24}; \quad x_{22} \le x_{32} \right\}.$$

T is a translation of the zero row/column sum version of $F''(\Delta^*)$. Furthermore it is easy to check that x^* is an extreme point of $F''(\Delta^*)$ if and only if $\phi(x^*)$ is an extreme point of T. Now T is a (row/column permutation version of the) face of a transportation polytope. Transportation polytopes occur as the feasible regions of transportation problems and their structure is well known. In particular the following graph theoretic characterization of the extreme points of T will be required below (see Klee and Witzgall (1968)).

A <u>loop</u> of $x \in T$ is defined as a sequence of nonzero entries in x such that (1) the row and column indices change alternately in the sequence and (2) the first and last elements of the sequence are the same but otherwise there is no repetition.

<u>Theorem 5.1</u>. A point $x \in T$ is a vertex if and only if x contains no loops.

Furthermore T is of dimension 5 in the 3×4 case and hence every vertex of T must contain 5 zeroes in addition to $x_{11} = 0$. There are 462 possible ways of placing the 5 zeroes among the 11 remaining positions and a systematic search through these candidates shows that the following elements of T contain no loops:

$$\begin{pmatrix} 0 & 3(\Delta^*-1) & 3(\Delta^*-1) & 3-2\Delta^* \\ 4\Delta^*-4 & 0 & 0 & 0 \\ 2-\Delta^* & 0 & 0 & 5\Delta^*-6 \end{pmatrix}, \quad 1.2 < \Delta^* \leq 1.5 \tag{5.3}$$

$$\begin{pmatrix} 0 & 3(\Delta^*-1) & 0 & \Delta^* \\ 4(\Delta^*-1) & 0 & 0 & 0 \\ \Delta^*-1 & 0 & 3(\Delta^*-1) & 2\Delta^*-3 \end{pmatrix}, \quad 1.5 \leq \Delta \leq 2 \tag{5.4}$$

$$(5.5)\qquad \begin{pmatrix} 0 & 3(\Delta^*-1) & 0 & \Delta^* \\ 2\Delta^*-1 & 0 & 0 & 2\Delta^*-3 \\ \Delta^*-1 & 0 & 3(\Delta^*-1) & 0 \end{pmatrix}, \quad \Delta^* \geq 1.5$$

$$(5.6)\qquad \begin{pmatrix} 0 & 3\Delta^*-3 & \Delta^* & 0 \\ 3\Delta^*-2 & 0 & 0 & \Delta^*-2 \\ 0 & 0 & 2\Delta^*-3 & 2\Delta^*-1 \end{pmatrix}, \quad \Delta^* \geq 2$$

$$(5.7)\qquad \begin{pmatrix} 0 & 3(\Delta^*-1) & 0 & \Delta^* \\ 3\Delta^*-2 & 0 & 0 & \Delta^*-2 \\ 0 & 0 & 3\Delta^*-3 & \Delta^*-1 \end{pmatrix}, \quad \Delta^* \geq 2 \text{ and}$$

$$(5.8)\qquad \begin{pmatrix} 0 & 2\Delta^*-1 & 2\Delta^*-2 & 0 \\ 0 & 0 & \Delta^*-1 & 3\Delta^* \\ 3\Delta^*-2 & \Delta^*-2 & 0 & 0 \end{pmatrix}, \quad \Delta^* \geq 2.$$

The complete set of vertices of $F'(\Delta^*)$ consists of (a) those point(s) (5.1) and/or (5.2) which are in $F'(\Delta^*)$ together with (b) the inverse images under ϕ of those points (5.3)-(5.8) satisfying the Δ^* condition. The results for general $F(2,3,\delta^*,\Delta^*)$ can easily be obtained from those for $F'(\Delta)$. The full set of extreme points of $F'(\Delta)$ are stated below in their familiar 3×4 form where rows and columns sum to zero:

I. <u>$1.2\delta^* < \Delta^* < 1.5\delta^*$:</u> LFC is one of

$$\begin{pmatrix} 2\delta^* & -2\delta^* & \delta^* & -\delta^* \\ -3\delta^* & \delta^* & \delta^* & \delta^* \\ \delta^* & \delta^* & -2\delta^* & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1.5\delta^* & -\delta^* & -\delta^* & .5\delta^* \\ -1.5\delta^* & .5\delta^* & .5\delta^* & .5\delta^* \\ 0 & .5\delta^* & .5\delta^* & -\delta^* \end{pmatrix} \text{ or}$$

$$\begin{pmatrix} \Delta^* & 2\delta^*-2\Delta^* & 2\delta^*-2\Delta^* & 3\Delta^*-4\delta^* \\ 3\delta^*-3\Delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* \\ 2\Delta^*-3\delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & 5\delta^*-4\Delta^* \end{pmatrix};$$

II. $\underline{\Delta^* = 1.5\delta^*}$: LFC is one of

$$\begin{pmatrix} 2\delta^* & -2\delta^* & \delta^* & -\delta^* \\ -3\delta^* & \delta & \delta & \delta \\ \delta^* & \delta^* & -2\delta^* & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1.5\delta^* & -\delta^* & -\delta^* & .5\delta^* \\ -1.5\delta^* & .5\delta^* & .5\delta^* & .5\delta^* \\ 0 & .5\delta^* & .5\delta^* & -\delta^* \end{pmatrix};$$

III. $\underline{1.5\delta^* < \Delta^* < 2\delta^*}$: LFC is one of

$$\begin{pmatrix} 2\delta^* & -2\delta^* & \delta^* & -\delta^* \\ -3\delta^* & \delta^* & \delta^* & \delta^* \\ \delta^* & \delta^* & -2\delta^* & 0 \end{pmatrix} \text{or} \begin{pmatrix} \Delta^* & 2\delta^*-2\Delta^* & \Delta^*-\delta^* & -\delta^* \\ -\Delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & 2\delta^*-\Delta^* \\ 0 & \Delta^*-\delta^* & 2\delta^*-2\Delta^* & \Delta^*-\delta^* \end{pmatrix} \text{or}$$

$$\begin{pmatrix} \Delta^* & 2\delta^*-2\Delta^* & \Delta^*-\delta^* & -\delta^* \\ 3\delta^*-3\Delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* \\ 2\Delta^*-3\delta^* & \Delta-\delta^* & 2\delta^*-2\Delta^* & 2\delta^*-\Delta^* \end{pmatrix};$$

IV. $\underline{\Delta^* = 2\delta^*}$: LFC is one of

$$\begin{pmatrix} 2\delta^* & -2\delta^* & \delta^* & -\delta^* \\ -2\delta^* & \delta^* & \delta^* & 0 \\ 0 & \delta^* & -2\delta^* & \delta^* \end{pmatrix} \text{or} \begin{pmatrix} 2\delta^* & -2\delta^* & \delta^* & -\delta^* \\ -3\delta^* & \delta^* & \delta^* & \delta^* \\ \delta^* & \delta^* & -2\delta^* & 0 \end{pmatrix};$$

V. $\underline{\Delta^* > 2\delta^*}$: LFC is one of

$$\begin{pmatrix} \Delta^* & 2\delta^*-2\Delta^* & \Delta^*-\delta^* & -\delta^* \\ 3\delta^*-3\Delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* \\ 2\Delta^*-3\delta^* & \Delta^*-\delta^* & 2\delta^*-2\Delta^* & 2\delta^*-\Delta^* \end{pmatrix} \text{or}$$

$$\begin{pmatrix} \Delta^* & 2\delta^*-2\Delta^* & -\delta^* & \Delta^*-\delta^* \\ \delta^*-2\Delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & \delta^* \\ \Delta^*-\delta^* & \Delta^*-\delta^* & 2\delta^*-\Delta^* & -\Delta^* \end{pmatrix} \text{ or}$$

$$\begin{pmatrix} \Delta^* & 2\delta^*-2\Delta^* & \Delta^*-\delta^* & -\delta^* \\ \delta^*-2\Delta^* & \Delta^*-\delta^* & \Delta^*-\delta^* & \delta^* \\ \Delta^*-\delta^* & \Delta^*-\delta^* & 2\delta^*-2\Delta^* & 0 \end{pmatrix} \text{ or}$$

$$\begin{pmatrix} \Delta^* & \delta^*-\Delta^* & -\Delta^* & \Delta^*-\delta^* \\ \Delta^*-\delta^* & 0 & \Delta^*-\delta^* & 2\delta^*-2\Delta^* \\ \delta^*-2\Delta^* & \Delta^*-\delta^* & \delta^* & \Delta^*-\delta^* \end{pmatrix}.$$

After making the change of variables described in Section 4 it is easy to check that $f_n(\underset{\sim}{\gamma})$ can be written in the form $P[A(P\underset{\sim}{X} + \sqrt{12n}/\sigma\underset{\sim}{\gamma}) \geq 0]$ where $X' = (X_{22},X_{23},X_{24},X_{32},X_{33},X_{34}) \sim N_6[\underset{\sim}{0},I_6]$, $\underset{\sim}{\gamma}' = (\gamma_{22},\gamma_{23},\gamma_{24},\gamma_{32},\gamma_{33},\gamma_{34})$, P is an orthogonal matrix satisfying

$$P'P = \begin{pmatrix} 6 & -2 & -2 & -3 & 1 & 1 \\ -2 & 6 & -2 & 1 & -3 & 1 \\ -2 & -2 & 6 & 1 & 1 & -3 \\ -3 & 1 & 1 & 6 & -2 & -2 \\ 1 & -3 & 1 & -2 & 6 & -2 \\ 1 & 1 & -3 & -2 & -2 & 6 \end{pmatrix} \text{ and}$$

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

For any fixed $R \equiv \Delta^*/\delta^* > 1.2$ the extreme points of $F'(2,3,\delta^*,\Delta^*)$ listed in I-V can each be written in the form $\delta^*\cdot\underset{\sim}{\nu}(R)$ where $\underset{\sim}{\nu}(R)$ is a 6×1 vector whose value is independent of δ^*. For example the three extreme points of I are: $\delta^*\cdot(1,1,1,1,-2,0)$, $\delta^*\cdot(.5,.5,.5,.5,.5,-1)$ and $\delta^*\cdot(R-1,R-1,R-1,R-1,R-1,5-4R)$ respectively. Hence for fixed R the PCS evaluated at extreme points $\delta^*\cdot\underset{\sim}{\nu}(R)$ is $f(\delta^*\cdot\underset{\sim}{\nu}(R)) = P[A(P\underset{\sim}{X} + \sqrt{12n}\ \delta^*/\sigma\underset{\sim}{\nu}(R)) \geq 0]$ which is a function of the scalar quantity $\sqrt{n}\ \delta^*/\sigma$. A short table of $\sqrt{n}\ \delta^*/\sigma$ values was constructed for various P^* and R pairs $(1.2 < R < 1.5)$ by (a) determining the zero of $h(\sqrt{n}\ \delta^*/\sigma) \equiv f_n(\delta^*\cdot\underset{\sim}{\nu}(R)) - P^*$ for each candidates LFC $\delta^*\cdot\underset{\sim}{\nu}(R)$ listed in I and (b) choosing as the true LFC that point having the largest $\sqrt{n}\ \delta^*/\sigma$ value associated with it. That largest $\sqrt{n}\ \delta^*/\sigma$ value was recorded in the table for the P^*, R coordinates. The values of $P[A(P\underset{\sim}{X} + \sqrt{12n}\ \delta^*/\sigma\cdot\underset{\sim}{\nu}(R)) \geq 0]$ where approximated by Monte Carlo simulation; vectors of six iid $N(0,1)$ random variables where generated and the set of inequalities in the above event were tested yielding a Bernoulli trial with the above "success" probability. The zero of $h(\sqrt{n}\ \delta^*/\sigma)$ was evaluated by a Robbins-Monro type of stochastic approximation scheme.

It was determined that for fixed $\sqrt{n}\ \delta^*/\sigma$, $P[A(P\underset{\sim}{X} + \sqrt{12n}\ \delta^*/\sigma\cdot\underset{\sim}{\nu}(R)) \geq 0]$ is extremely flat in R over (1.2,1.5); furthermore the same point,

$\delta^*\cdot(1,1,1,1,-2,0)$ was the LFC in all cases studied. Hence the following table gives values of $\sqrt{n}\,\delta^*/\sigma$ which are accurate for the entire range $1.2 < R = \Delta^*/\delta^* < 1.5$ for each listed P^* value.

P^*	.90	.95	.99
$\dfrac{\sqrt{n}\,\delta^*}{\sigma}$	1.99	2.35	2.66

The values listed have been rounded off from 3 place computations and should be accurate to two places. These computations were carried out on Cornell University's IBM 360/168 computer.

6. Preferred Population Formulation

This section describes the preferred population formulation of Fabian (1962) for determining the sample size to be used with procedure P. This formulation is a strengthening of the indifference zone requirement (2.1) (see Fabian (1962) and Panchapakesan and Santner (1977)).

Fix δ^* and Δ^* as in Section 2 and call the population interaction γ_{ij} (or treatment combination (i,j)) preferred (or near optimal) iff either (a) $\gamma_{[rc]} < \Delta^*$ or (b) $\gamma_{[rc]} \geq \Delta^*$ and $\gamma_{ij} > \gamma_{[rc]} - \delta^*$. The goal is to select any treatment combination having a near optimal γ_{ij}. Let Ω be the space of all $r \times c$ matrices with row and column sums equal to zero and $P^* \in (1/rc, 1)$. The probability (design) requirement to be guaranteed is

$$P_{\underset{\sim}{\gamma}}[CS|P] \geq P^* \ \forall \ \underset{\sim}{\gamma} \in \Omega \tag{6.1}$$

where the event $[CS|P]$ occurs iff a preferred treatment combination is selected.

Let Ω_ℓ, $1 \leq \ell \leq rc$ be that subset of Ω in which exactly ℓ γ_{ij}'s are preferred. Some Ω_i's may be empty, however it can easily be shown that $\Omega_{rc} = \{\underset{\sim}{\gamma} \in \Omega | \gamma_{[rc]} < \Delta^*\}$ and $\Omega_1 = \{\underset{\sim}{\gamma} \in \Omega | \gamma_{[rc]} \geq \Delta^*;$ $\gamma_{[rc]} - \gamma_{[rc-1]} \geq \delta^*\}$. Since $P_{\underset{\sim}{\gamma}}[CS|P] = 1 \ \forall \ \underset{\sim}{\gamma} \in \Omega_{rc}$ it suffices to compute $\inf_{\Omega_i} P_{\underset{\sim}{\gamma}}[CS|P]$ for $i \in I \equiv \{1,\ldots,rc-1\}$ in order to determine $\inf_{\Omega} P_{\underset{\sim}{\gamma}}[CS|P]$. For $i \in I$ define

$$E_i(\underset{\sim}{\gamma}) = [\max\{X_{(rc)}+\gamma_{[rc]},\ldots,X_{(rc-i+1)}+\gamma_{[rc-i+1]}\}$$

$$> \max\{X_{(rc-i)}+\gamma_{[rc-i]},\ldots,X_{(1)}+\gamma_{[1]}\}]$$

where $X_{(\ell)} = \hat{Y}_{(\ell)} - \gamma_{[\ell]}$ and $\hat{Y}_{(\ell)}$ is the sample interaction having mean $\gamma_{[\ell]}$. Note that $P_{\underset{\sim}{\gamma}}[CS|P] = P[E_i(\underset{\sim}{\gamma})]\ \forall\ \underset{\sim}{\gamma} \in \Omega_i$ and the events $E_i(\underset{\sim}{\gamma})$ are nondecreasing in i for fixed $\underset{\sim}{\gamma}$.

<u>Remark 6.1</u>. Requirement (6.1) implies that $P[E_1(\underset{\sim}{\gamma})] \geq P^* \ \forall\ \underset{\sim}{\gamma} \in \Omega_1$ which is (2.1) and hence (6.1) is a strengthening of the indifference zone approach.

<u>Theorem 6.1</u>. $\inf_{\Omega} P_{\underset{\sim}{\gamma}}[CS|P] = \min_{i \in I} \inf_{\Omega_i} P_{\underset{\sim}{\gamma}}[E_i(\underset{\sim}{\gamma})] = \inf_{\Omega_1} P[E_1(\underset{\sim}{\gamma})]$ and hence the same sample size achieves both (2.1) and (6.1).

<u>Proof</u>. Given i, $2 \leq i < rc$, it follows that the ordered components of $\underset{\sim}{\gamma} \in \Omega_i$ satisfy $\gamma_{[rc]} \geq \Delta^*$ and $\gamma_{[rc]} \geq \cdots \geq \gamma_{[rc-i+1]} > \gamma_{[rc]}-\delta^* \geq \gamma_{[rc-i]} \geq \cdots \geq \gamma_{[1]}$. By decreasing $\gamma_{[rc-i+1]}$ to $\gamma'_{[rc-i+1]} \leq \gamma_{[rc]}-\delta^*$ and increasing the components $\gamma_{[1]},\ldots,\gamma_{[rc-i]}$ to preserve row and column sums equal to zero it can be seen that there exists $\underset{\sim}{\gamma}' = \underset{\sim}{\gamma}'(\underset{\sim}{\gamma}) \in \Omega_{i-1}$ satisfying $\gamma_{[\ell]} \geq \gamma'_{[\ell]}$, $rc-i+1 \leq \ell \leq rc$ and $\gamma_{[\ell]} \leq \gamma'_{[\ell]}$, $1 \leq \ell \leq rc-i$. This implies $E_i(\underset{\sim}{\gamma}) \supset E_i(\underset{\sim}{\gamma}')$ and hence $\forall\ \underset{\sim}{\gamma} \in \Omega_i$

$$
\begin{aligned}
P[E_i(\underset{\sim}{\gamma})] &\geq P[E_i(\underset{\sim}{\gamma}')] \\
&\geq P[E_{i-1}(\underset{\sim}{\gamma}')] \quad \text{since } E_{i-1}(\underset{\sim}{\gamma}') \subset E_i(\underset{\sim}{\gamma}') \\
&\geq \inf_{\underset{\sim}{\Gamma} \in \Omega_{i-1}} P[E_{i-1}(\underset{\sim}{\Gamma})] \quad \text{since } \underset{\sim}{\gamma}' \in \Omega_{i-1}
\end{aligned}
$$

$\Rightarrow \inf_{\Omega_i} P[E_i(\underset{\sim}{\Gamma})] \geq \inf_{\Omega_{i-1}} P[E_{i-1}(\underset{\sim}{\Gamma})]$, $2 \leq i < rc$ and the result follows by induction.

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The problem of devising a single-stage procedure for selecting the factor-level combination associated with the largest positive interaction is studied for the two-factor $r \times c$ ($r \geq 2$, $c \geq 3$) experiment involving qualitative variables when the common variance is known. The intuitive procedure based on the best linear unbiased estimators of the population → next page

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interactions is employed.

Initially the problem is formulated using an indifference zone approach; the experimenter is required to specify quantities $(\Delta^*, \delta^*, P^*)$ satisfying $0 < \delta^*$, $\delta^* \cdot \frac{(r-1)(r-1)}{(r-1)(c-1)-1} < \Delta^*$ and $\frac{1}{rc} < P^* < 1$ prior to the start of experimentation. These quantities are incorporated into a probability (design) requirement which must be satisfied by the selection procedure. The paper analyzes the LFC based on the log-concavity of the PCS regarded as a function of the population interactions and on the characteristics of the preference zone. The 3×4 case is examined in detail.

The problem is reformulated using a preferred population approach which employs a strengthened version of the indifference zone probability requirement. It is shown that the same sample size guarantees this strengthened probability requirement as does the earlier one.